Exact Solutions to the Oscillations of Composite Aircraft Wings with Warping Constraint and Elastic Coupling

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Exact solutions within the framework of standard aeroelastic bending and twisting assumptions are found to the free oscillations of composite aircraft wings with warping constraint and elastic coupling. The problem is treated as a regular boundary-value problem consisting of two fourth-order partial-differential equations coupled by the presence of elastic coupling. This system, which is linear, is therefore equivalent to an eighth-order ordinary-differential equation. Classical linear "operator" methods are used to extract fundamental solutions that are superimposed appropriately to obtain an exact functional form for the mode shapes. These mode shapes are then required to satisfy the necessary boundary conditions, a process that leads to the formulation of the required eigenvalue problem. The eigenvalues are extracted numerically by using appropriate ordering of the eight roots of the operator equation. The bending-torsion frequencies obtained as a result of this analysis are compared favorably with existing results. New insights made possible by these results, which are preliminary, appear to be that 1) the first coupled frequency decreases with increasing coupling, and 2) the phenomenon of modal transformations found by earlier investigators is explainable in terms of some conservative intermodal energy transfer.

Nomenclature

a_i	= chordwise integrals
c_0', c_0	= affine space half-chord and chord, respectively
D_{ij}	= elastic constants
$D^*, D_0^{*\prime}$	= generic nondimensionalized stiffness parameters
e	= parameter that measures the location of the reference axis relative to midchord
$(\bar{h}, h_0)$	= wing box depth and affine space bending displacement, respectively
$\bar{k}_n$	= nondimensionalized frequency parameter
ℓ_0	= affine space half-span for the wing
L_1, L_2	= generic nondimensionalized stiffness parameters
L_0, M_0	= affine space running aerodynamic lift and moments, respectively
m_0	= affine space mass per unit span
$(\Delta P, \Delta P_0)$	= differential aerodynamic pressure distributions in physical and affine space, respectively
r, t	= generic nondimensionalized stiffness parameter and time, respectively
U, U_0	= virtual work expressions in physical and affine space, respectively
w	= displacement
$(x, y, z,), (x_0, y_0, z_0)$	= physical and affine space coordinates, respectively
α_0	= affine space torsional displacement
ρ, ρ_∞	= affine space material and air density, respectively
ω_n	= vibration frequency

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Introduction

PERHAPS one of the more elusive aspects of supermaneuverability as a design concept is its aeroelastic implications. One generally accepted definition of a supermaneuverable aircraft is that it is designed to operate at high angles of attack. Strictly speaking, high-angle-of-attack problems are nonlinear. However, due to the high degree of complexities involved in dealing with nonlinear aeroelastic problems, an aeroelastician would prefer to deal with a linearized version of the problem (at least as a first approximation). If linear aeroelastic equations are used under such conditions, at least it should be assumed that the high angle of attack would introduce large twisting displacements, which would imply that terms containing twisting displacement should be retained. Even under low-angle-of-attack assumptions, the early works of Reissner and Stein[1] and later works of Librescu et al.[2] have shown that for metal wings, there are conditions under which the so-called St. Venant's torsion principle is inapplicable. This is when the restraint of the warping (an assumption that plane sections remain plane during deformation) effect is important, and a more accurate analysis would need to include a higher-order term involving the twisting displacement. Although the retention of such a term implies solving fourth-order (instead of second-order) differential equations for the twisting and bending displacements, the equations can easily be decoupled for metal wings. However, for composite wings, the decoupling of these equations is neither easy practically nor even desirable from an aeroelastic tailoring standpoint.[3-7] These studies have also shown that the restraint of warping is very important in composite wings. Therefore, it would seem that an aeroelastic analysis of a supermaneuverable (high-angle-of-attack) aircraft wing fabricated of composite materials would need to consider the effects of restraint of warping as well as elastic coupling.

Previous investigation of this latter problem (free vibration) at MIT[3,4] used analytical methods to solve the decoupled problem, while numerical methods were utilized to solve the coupled problems. Consequently, general results were presented for the decoupled problem, while representative results were presented for the coupled problem.

In this paper the coupled free vibration is treated analytically as a pair of coupled fourth-order differential equations (a boundary-value problem) to which exact closed-form eigensolutions are sought. The enforcement of the necessary boundary conditions resulted in a fairly complicated transcendental function to be used to determine the required eigenvalues from which the natural frequencies are to be obtained. This transcendental function was complex in contrast to its decoupled counterpart, which is real. That should indicate the presence of the phase angle that exists between the twisting and bending displacements. A comparison with a damped (decoupled) system in which a complex determinant signifies phase angles between damping and other forces led us to the formulation of an explanation for the "modal transformation" phenomenon, which was reported in studies at MIT[3,4] and Purdue[8] (and which seemed to have lacked explanation until this study). The explanation is that the modal transformation may be viewed as a form of steady-state conservative intermodal energy transfer between the vibration modes (energy stays in this system since there is no damping). In fact, work currently in progress at Purdue[8] seems to confirm this explanation. The results, which compared favorably with those obtained at MIT,[3,4] also revealed that coupling has a tendency to lower the first coupled natural frequency of a composite aircraft wing. In fact, it is seen that a substantial amount of coupling could reduce the first coupled frequency to almost zero (hence, a possibility of coupling with rigid-body modes).

Problem Statement

For a composite aircraft wing cantilevered at the root as shown in Fig. 1, the virtual work theorem in the physical space is given by

$$\delta\bar U = 0 = \frac{\delta}{2}\int_0^t\int_A\int [D_{11}(w_{,xx})^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}(w_{,yy})^2] + 4D_{16}w_{,xx}w_{,xy} + 4D_{26}w_{,yy}w_{,yy} + 4D_{66}(w_{,xy})^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \frac{\delta}{2}\int_0^t\int_A\int \rho\bar h\dot w^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_0^t\int_A\int \Delta p\,\delta w\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \tag{1}$$

Using the following affine transformation of variables,

$$x = \left(\frac{D_{11}}{D_{22}}\right)^{1/4} x_0, \qquad y = y_0, \qquad z = z_0 \tag{2a}$$

Then in affine space the virtual work theorem becomes

$$\delta\bar U_0 = 0 = 2\int_0^t\int_A\int \{(w_{,x_0x_0})^2 + 2D^*[(1-\epsilon)(w_{,x_0y_0})^2 + \epsilon w_{,x_0x_0}w_{,y_0y_0}] + (w_{,y_0y_0})^2 + L_1 w_{,x_0x_0}w_{x_0y_0} + L_2 w_{,y_0y_0}w_{,x_0y_0}\}\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}t - \frac{\delta}{2}\int_0^t\int_A\int \rho_0\dot w^2\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}t + \int_0^t\int_A\int \Delta p_0\delta w\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}t \tag{2b}$$

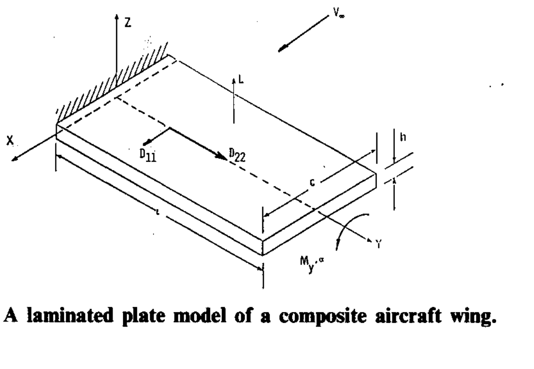

Fig. 1 A laminated plate model of a composite aircraft wing.

where

$$\bar U_0 = \frac{\bar U}{D_{22}}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}, \qquad D^* = \frac{D_{12}+2D_{66}}{(D_{11}D_{22})^{1/2}} \tag{3a}$$

$$\epsilon D^* = \frac{D_{12}}{(D_{11}D_{22})^{1/2}}, \qquad L_1 = \frac{4D_{16}}{(D_{11})^{3/4}(D_{22})^{1/4}} \tag{3b}$$

$$L_2 = \frac{4D_{26}}{(D_{11})^{1/4}(D_{22})^{3/4}}, \qquad \Delta p_0 = \frac{\Delta p}{D_{22}}, \qquad \rho_0 = \frac{\rho\bar h}{D_{22}} \tag{3c}$$

If the affine space equivalent of the standard aeroelastic displacement assumptions is made, i.e.,

$$w(x_0,y_0,t) = h_0(y_0,t) + x_0\alpha_0(y_0,t)\ldots \tag{4}$$

where h_0 and α_0 are the bending and twisting displacements, respectively, then it can be shown through the use of the calculus of variations that a coupled set of aeroelastic equations of motion for the composite aircraft wing in which the restraint of warping and elastic coupling effects are accounted for is given by

$$a_1h_0^{iv} + a_2\alpha_0^{iv} - a_5\alpha_0^{iii} + \rho_0a_1\ddot h_0 + \rho_0a_2\ddot\alpha_0 = L_0 \tag{5a}$$

$$a_2h_0^{iv} + a_5h_0^{iii} + a_3\alpha_0^{iv} - a_4\alpha_0'' + \rho_0a_3\ddot\alpha_0 + \rho_0a_2\ddot h_0 = M_0 \tag{5b}$$

with boundary conditions

at $y_0 = 0$

$$h_0 = 0, \qquad h_0' = 0, \qquad \alpha_0 = 0, \qquad \alpha_0' = 0 \tag{6a}$$

at $y_0 = \ell_0$

$$a_3\alpha_0'' + a_2h_0'' - L_2a_2\alpha_0' = 0, \qquad a_1h_0'' - a_5\alpha_0' + a_2\alpha_0'' = 0 \tag{6b}$$

$$a_2\alpha_0''' + a_1h_0''' - a_5\alpha_0'' = 0 \tag{6c}$$

$$a_2h_0''' + a_3\alpha_0''' + a_5h_0'' - a_4\alpha_0' = 0 \tag{6d}$$

where

$$a_1 = \int_{e\bar c_0}^{\bar c_0}\mathrm{d}x_0, \qquad a_2 = \int_{e\bar c_0}^{\bar c_0}x_0\,\mathrm{d}x_0 \tag{7a}$$

$$a_3 = \int_{e\bar c_0}^{\bar c_0}x_0^2\,\mathrm{d}x_0, \qquad a_4 = 2\int_{e\bar c_0}^{\bar c_0}D^*(1-\epsilon)\,\mathrm{d}x_0 \tag{7b}$$

$$L_0 = \int_{e\bar c_0}^{\bar c_0}\Delta p_0\,\mathrm{d}x_0, \qquad a_5 = \int_{e\bar c_0}^{\bar c_0}L_2\,\mathrm{d}x_0 \tag{7c}$$

$$M_0 = \int_{e\bar c_0}^{\bar c_0}x_0\,\Delta p_0\,\mathrm{d}x_0 \tag{7d}$$

$$-\infty < e < 0, \qquad \bar c_0 = \frac{c_0}{1-e} \tag{7e}$$

$$(\ )' = \frac{\partial}{\partial y_0}, \qquad (\dot{\ }) = \frac{\partial}{\partial t} \tag{7f}$$

For free vibrations, if a_2 (through the geometric construction of the wing) is made to be zero, Eqs. (5) reduce to

$$a_1h_0^{iv} - a_5\alpha_0''' + \rho_0a_1\ddot h_0 = L_0 = 0$$

$$a_3\alpha_0^{iv} + a_5h_0''' - a_4\alpha'' + \rho_0a_3\ddot\alpha_0 = M_0 = 0 \tag{8a}$$

with boundary conditions

at $y = 0$

$$h_0 = 0, \qquad h_0' = 0, \qquad \alpha_0 = 0, \qquad \alpha' = 0$$

at $y_0 = \ell_0$

$$a_1 h_0''' = a_5 \alpha_0''' = 0, \qquad a_1 h_0'' - a_5 \alpha_0' = 0$$

$$a_3 \alpha_0''' + a_5 h_0''' - a_4 \alpha_0' = 0 \tag{8b}$$

It may be stated here that the restraint of the warping effect is represented by the product of α^{IV} and a_3, while the elastic coupling effect is represented by the a_5 terms.

Methods of Solution

Two methods for solving Eqs. (8) are examined in this paper. These are 1) an "exact closed-form" approach, and 2) a "semiexact closed-form" approach. The exact closed-form approach is defined here as one in which explicit expressions are derived for the eight roots of the eighth-order operator equations representing Eqs. (8a), and through the superimposition of fundamental solutions corresponding to each of the eight operator roots, the boundary conditions of Eqs. (8b) are satisfied. The semiexact closed-form approach is the same as the exact, except that the roots of the operator equation are determined numerically through some standard root extraction subroutines.

In either case, to solve for the operator roots of Eqs. (8a), they may be rewritten in operator form as follows:

$$\bar{L}_1 \bar{h}_0 - \bar{L}_2 \bar{\alpha}_0 = 0 \tag{9a}$$

$$\bar{L}_2 \bar{h}_0 + \bar{L}_3 \bar{\alpha}_0 = 0 \tag{9b}$$

where the operators $\bar{L}_n$ ($n = 1,2,3$) are given by

$$\bar{L}_1 = a_1 \frac{\partial^4}{\partial \bar{y}_0^4} - a_1 \omega^2 \rho_0, \qquad \bar{L}_2 = a_5 \frac{\partial^3}{\partial \bar{y}_0^3} \tag{10a}$$

$$\bar{L}_3 = a_3 \frac{\partial^4}{\partial \bar{y}_0^4} - a_4 \frac{\partial^2}{\partial \bar{y}_0^2} - a_3 \omega^2 \rho_0 \tag{10b}$$

and

$$h_0 = \bar{h}_0 e^{i\omega t} \tag{10c}$$

$$\alpha_0 = \bar{\alpha}_0 e^{i\omega t} \tag{10d}$$

where ω is vibration frequency. Following the method used in Hildebrand,[9] if $\bar{L}_1$ and $\bar{L}_2$ are commutative (i.e., $\bar{L}_1\bar{L}_2 = \bar{L}_2\bar{L}_1$) or $\bar{L}_2$ and $\bar{L}_3$ are commutative (i.e., $\bar{L}_2\bar{L}_3 = \bar{L}_3\bar{L}_2$), either $\bar{h}_0$ or $\bar{\alpha}_0$ may be eliminated and Eqs. (9) are reduced to one eighth-order operator equation in either $\bar{\alpha}_0$ or $\bar{h}_0$, respectively. This can be accomplished by using $\bar{L}_2$ to operate on Eq. (9a) and $\bar{L}_1$ to operate on Eq. (9b) and subtracting Eq. (9a) from Eq. (9b), or by using $\bar{L}_3$ to operate on Eq. (9a) and $\bar{L}_2$ to operate on Eq. (9b) and subtracting Eq. (9a) from Eq. (9b). The result of such an operation in which $\bar{h}_0$ is eliminated is given by

$$(\bar{x}^4 + F_3\bar{x}^3 + F_2\bar{x}^2 + F_1\bar{x} + F_0)\alpha_0 = 0$$

where

$$F_3 = -16\bar{\lambda}_c^2, \qquad F_2 = -\bar{\kappa}^2, \qquad F_1 = 8(\bar{\kappa}\lambda_c)^2$$

$$F_0 = \frac{\bar{\kappa}^4}{4}, \qquad \bar{x} = \left(\frac{\partial}{\partial y_0}\right)^2 \tag{11}$$

It should be noted here that because the method of elimination of $\bar{h}_0$ is through a differential operator, superfluous solutions are to be expected [e.g., operator (11) is eighth-order instead of fourth-order]. These superfluous solutions are eliminated by enforcing some consistency conditions on the solutions as will be described. Based on Eqs. (11), an appropriate form for the exact closed-form (or complete) solution for the mode shapes of the free vibration problem proposed in this paper, which can also be made to satisfy the necessary boundary conditions, is given by

$$\begin{aligned} h_0 = {} & A_1 \cosh\beta_1 y_0 + A_2 \sinh\beta_1 y_0 + A_3 \cos\beta_2 y_0 \\ & + A_4 \sin\beta_2 y_0 + A_5 \cosh\beta_3 y_0 + A_6 \sinh\beta_3 y_0 \\ & + A_7 \cos\beta_4 y_0 + A_8 \sin\beta_4 y_0 \end{aligned} \tag{12a}$$

$$\begin{aligned} \alpha_0 = {} & B_1 \cosh\beta_1 y_0 + B_2 \sinh\beta_1 y_0 + B_3 \cos\beta_2 y_0 \\ & + B_4 \sin\beta_2 y_0 + B_5 \cosh\beta_3 y_0 + B_6 \sinh\beta_3 y_0 \\ & + B_7 \cos\beta_4 y_0 + B_8 \sin\beta_4 y_0 \end{aligned} \tag{12b}$$

where

$$\beta_n = \bar{x}_n^{1/2}, \qquad n = 1,2,3,4 \tag{13}$$

and $\bar{x}_n$ can be obtained either from the numerical solution of Eqs. (11) or according to the method described in Abramowitz and Stegun,[10] as follows. Define

$$\lambda_c = (\ell_0/c_0)\sqrt{\frac{3}{2}D_0^*}, \qquad \bar{\lambda}_c = (\ell_0/c_0)\sqrt{\frac{3}{2}\left(D_0^* - \frac{L_2^2}{2}\right)} \tag{14}$$

$$P_1 = 4(\bar{\lambda}_c^4 + \lambda_c^4) - 8/3\bar{\lambda}_c^2\lambda_c^2 + \frac{\bar{\kappa}^2}{27} \tag{15a}$$

$$P_2 = \frac{1}{6}\left(32\bar{\lambda}_c^2\lambda_c^2 + \frac{\bar{\kappa}^2}{3}\right) \tag{15b}$$

$$S_1 = 2\bar{\kappa}[\bar{\kappa}P_1 + (\bar{\kappa}^2 P_1^2 - 8P_2^3)^{1/2}]^{1/3} \tag{16a}$$

$$S_2 = 2\bar{\kappa}[\bar{\kappa}P_1 - (\bar{\kappa}^2 P_1^2 - 8P_2^3)^{1/2}]^{1/3} \tag{16b}$$

$$u_1 = S_1 + S_2 - \frac{\bar{\kappa}^2}{3} \tag{17a}$$

$$u_2 = -\frac{1}{2}(S_1 + S_2) - \frac{\bar{\kappa}^2}{3} + i\frac{\sqrt{3}}{2}(S_1 - S_2) \tag{17b}$$

$$u_3 = -\frac{1}{2}(S_1 + S_2) - \frac{\bar{\kappa}^2}{3} + i\frac{\sqrt{3}}{2}(S_1 - S_2) \tag{17c}$$

$$f_{3,4} = -8\bar{\lambda}_c^2 \mp [64\bar{\lambda}_c^4 + u_i + \bar{\kappa}^2]^{1/2} \tag{18a}$$

$$f_{5,6} = \frac{u_i}{2} \pm \left[\left(\frac{u_i}{2}\right)^2 - \frac{\bar{\kappa}^4}{4}\right]^{1/2} \tag{18b}$$

$$\bar{x}_{1,2} = \frac{-f_3 \pm \sqrt{f_3^2 - 4f_5}}{2}, \qquad \bar{x}_{3,4} + \frac{-f_4 \pm \sqrt{f_4^2 - 4f_6}}{2} \tag{19}$$

where u_i ($i = 1,2,3$) is the root that makes $f_{3,4,5,6}$ all real.

Consistency Conditions

From Eqs. (12a) and (12b) it is seen that there are 16 arbitrary integration constants as opposed to the expected eight constants. The additional eight constants have been introduced superfluously as a result of the differential operation, which was done in order to eliminate one of the dependent variables in the two coupled differential equations, Eqs. (9). In order to get rid of these superfluous solutions it is necessary to enforce some consistency conditions. This may be accomplished by substituting Eqs. (12a) and (12b) into one of

Eqs. (9) and requiring the equation to be satisfied identically. This procedure establishes a set of explicit relationships between the constants A_n and B_n in which A_n can be determined in terms of B_n or vice versa.

When the superfluous solutions are eliminated, Eqs. (12) can now be used to satisfy the boundary conditions for this problem [Eqs. (8b)]. Consequently, the condition for nontrivial solutions is enforced to obtain the transcendental functional expression for determining the eigenvalues of this problem.

Eigenvalues

The following steps and definitions are carried out in order to obtain the transcendental functional expression for determining the necessary eigenvalue (for this eighth-order boundary-value problem) from which the coupled natural frequencies may be obtained. Define

$$h_n = \beta_n^4 - \frac{\bar{k}^2}{2}, \qquad t_n = \beta_n^2 - \frac{1}{\beta_n^2} h_n \tag{20a}$$

$$\bar{h}_n = \frac{h_n}{\beta_n^3}, \qquad \hat{h}_n = \frac{h_n}{\beta_n} \tag{20b}$$

$$\bar{t}_n = 16(\lambda_c^2 - \bar{\lambda}_c^2)\beta_n^2 + (-1)^{n+1}\frac{h_n}{\beta_n^2}[\beta_n^2 + (-1)^n 16\lambda_c^2] \tag{21}$$

$$a_{15} = \frac{\bar{h}_3 + \bar{h}_2}{\bar{h}_1 + \bar{h}_2}, \qquad a_{17} = \frac{\bar{h}_4 - \bar{h}_2}{\bar{h}_1 + \bar{h}_2}, \qquad a_{35} = \frac{\bar{h}_3 - \bar{h}_1}{\bar{h}_1 + \bar{h}_2} \tag{22a}$$

$$a_{37} = \frac{\bar{h}_1 + \bar{h}_4}{\bar{h}_1 + \bar{h}_2}, \qquad a_{26} = \frac{\bar{h}_3 - \bar{h}_2}{\bar{h}_2 - \bar{h}_1}, \qquad a_{28} = \frac{\bar{h}_4 - \bar{h}_2}{\bar{h}_2 - \bar{h}_1} \tag{22b}$$

$$a_{46} = \frac{\bar{h}_1 - \bar{h}_3}{\bar{h}_2 - \bar{h}_1}, \qquad a_{48} = \frac{\bar{h}_1 - \bar{h}_4}{\bar{h}_2 - \bar{h}_1} \tag{22c}$$

$$F_{51} = -\beta_1^3 a_{15}\sinh\beta_1 + \beta_2^3 a_{35}\sin\beta_2 + \beta_3^3\sinh\beta_3 \tag{23a}$$

$$F_{61} = \beta_1^3 a_{26}\cosh\beta_1 - \beta_2^3 a_{46}\cos\beta_2 + \beta_3^3\cosh\beta_3 \tag{23b}$$

$$F_{71} = \beta_1^3 a_{17}\sinh\beta_1 - \beta_2^3 a_{37}\sin\beta_2 + \beta_4^3\sin\beta_4 \tag{23c}$$

$$F_{81} = \beta_1^3 a_{28}\cosh\beta_1 - \beta_2^3 a_{48}\cos\beta_2 - \beta_4^3\cos\beta_4 \tag{23d}$$

$$F_{52} = \hat{h}_1 a_{15}\sinh\beta_1 - \hat{h}_2 a_{35}\sin\beta_2 - \hat{h}_3\sinh\beta_3 \tag{23e}$$

$$F_{62} = -\hat{h}_1 a_{26}\cosh\beta_1 + \hat{h}_2 a_{46}\cos\beta_2 - \hat{h}_3\cosh\beta_3 \tag{23f}$$

$$F_{72} = -\hat{h}_1 a_{17}\sinh\beta_1 + \hat{h}_2 a_{37}\sin\beta_2 - \hat{h}_4\sin\beta_4 \tag{24a}$$

$$F_{82} = -\hat{h}_1 a_{28}\cosh\beta_1 + \hat{h}_2 a_{48}\cos\beta_2 + \hat{h}_4\cos\beta_4 \tag{24b}$$

$$F_{53} = -t_1 a_{15}\cosh\beta_1 - t_2 a_{35}\cos\beta_2 + t_3\cosh\beta_3 \tag{24c}$$

$$F_{63} = t_1 a_{26}\sinh\beta_1 - t_2 a_{46}\sin\beta_2 + t_3\sinh\beta_3 \tag{24d}$$

$$F_{73} = t_1 a_{17}\cosh\beta_1 + t_2 a_{37}\cos\beta_2 - t_4\cos\beta_4 \tag{24e}$$

$$F_{83} = t_1 a_{28}\sinh\beta_1 - t_2 a_{48}\sin\beta_2 - t_4\sin\beta_4 \tag{24f}$$

$$F_{54} = \bar{t}_1 a_{15}\cosh\beta_1 + \bar{t}_2 a_{35}\cos\beta_2 - \bar{t}_3\cosh\beta_3 \tag{24g}$$

$$F_{64} = -\bar{t}_1 a_{26}\sinh\beta_1 + \bar{t}_2 a_{46}\sin\beta_2 - t_3\sinh\beta_3 \tag{24h}$$

$$F_{74} = -\bar{t}_1 a_{17}\cos\beta_1 - \bar{t}_2 a_{37}\cos\beta_2 + \bar{t}_4\cos\beta_4 \tag{24i}$$

$$F_{84} = -\bar{t}_1 a_{28}\sinh\beta_1 + \bar{t}_2 a_{46}\sin\beta_2 + \bar{t}_4\sin\beta_4 \tag{24j}$$

$$a_{57} = \frac{F_{61}F_{72} - F_{62}F_{71}}{F_{51}F_{62} - F_{52}F_{61}}, \qquad a_{58} = \frac{F_{61}F_{82} - F_{62}F_{81}}{F_{51}F_{62} - F_{52}F_{61}} \tag{25a}$$

$$a_{67} = \frac{F_{52}F_{71} - F_{51}F_{72}}{F_{51}F_{62} - F_{52}F_{61}}, \qquad a_{68} = \frac{F_{52}F_{81} - F_{51}F_{82}}{F_{51}F_{62} - F_{52}F_{61}} \tag{25b}$$

Then $\bar{k}_n$ are given by the roots of

$$\bar{F} = (F_{53}a_{57} + F_{63}a_{67} + F_{73})(F_{54}a_{58} + F_{64}a_{68} + F_{84}) - (F_{54}a_{57} + F_{64}a_{67} + F_{74})(F_{53}a_{58} + F_{63}a_{68} + F_{83}) = 0 \tag{26}$$

Equation (26) is, therefore, the exact closed-form trancendental functional expression from which the eigenvalues $\bar{k}_n$ may be extracted. The coupled natural frequencies are related to these eigenvalues by the following expression:

$$\omega_n = \left(\frac{D_{22}}{\rho}\right)^{1/2}\frac{\bar{k}_n}{\ell_0^2}(\lambda_c, \bar{\lambda}_c/\lambda_c)\ldots \tag{27}$$

Computations

The extraction of the eigenvalues from the exact closed-form transcendental expression in Eq. (26) proved to be a very challenging computational exercise, due mainly to its complex nature, the existence of branch points, the necessity to order the operator roots appropriately, and the existence of numerical noise. The characteristic roots for the equations of motion, x_n, were extracted in two different ways in order to assure accuracy. One method was by using a Jenkins-Traub computational method from the IMSL library. The second method was by using an exact closed-form approach outlined in Abramowitz and Stegun.[10] When these roots are being extracted numerically, they do not necessarily come out in a continuous manner. Therefore, a subroutine that reorders them so that they become continuous with respect to the eigenvalue $\bar{k}_n$ is also employed. Once this reordering of these roots is completed, the parameters needed for computing the transcendental function $\bar{F}$ in Eq. (26) are computed. Finally, the transcendental expression itself is computed and the roots $\bar{k}_n$ are found numerically.

One interesting result is that the values of $\bar{F}$ are complex here. It is known, however, that each of the two uncoupled problems (bending or torsion) when treated separately has a real transcendental expression for extracting the eigenvalues. This behavior of the transcendental expression for extracting the coupled eigenvalues (i.e., being complex as opposed to being real) was the first hint that led the first author to examine any possible mathematical similarity between the problem at hand (a coupled nondamped oscillations problem) and a simple damped oscillations problem in which the expression for determining the eigenvalues is in general complex due to the need to determine the oscillation frequency and the amount of damping in the mode, etc. In the coupled problem the complex nature of this transcendental expression for extracting the eigenvalues seems to be basically a reflection of the phase that normally should exist between the bending and torsional modes. Realizing that in the damped case, there is a nonconservative energy transfer between the oscillating system and its environment as well as intermodal energy transfer, it became clear to the first author that in the undamped coupled system there may be a conservative energy transfer in the oscillating system. The presence of an odd derivative in Eqs. (8a) may suggest the potential for a damping-like behavior.[11] At the time of this thought it seemed to the first author that if it made sense, the idea should provide an explanation for the modal changes noticed during the studies at Purdue and MIT,[3,4] as coupling in the system was changed. This prediction, which turns out to be useful and is confirmed by other recent studies from Purdue University,[8] shall be discussed in more detail.

The most convenient way to obtain the eigenvalues $\bar{k}_n$ was found to be by graphical means. Thus, the values of the real and imaginary parts of the complex transcendental function $\bar{F}$ are plotted against the values of $\bar{k}_1$ on the same curve, as shown in Fig. 2. The values of $\bar{k}_n$ at which the real and imaginary parts of the transcendental function are simulta-

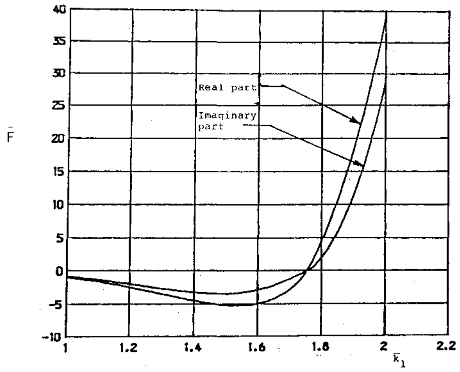

Fig. 2 Variation of the transcendental function $\bar{F}$ with the eigenvalue $\bar{k}_1$ ($\bar{\lambda}_c/\lambda_c = 0.638$, $\lambda_c = 0.732$).

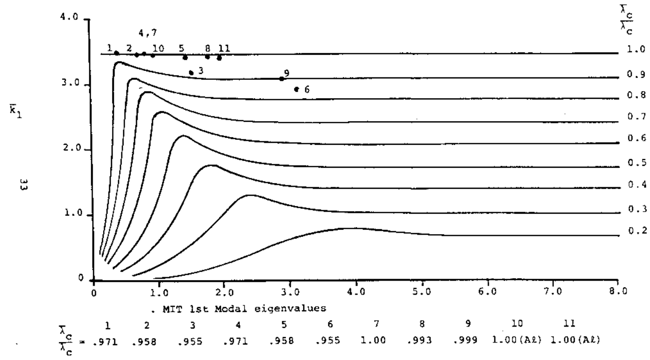

Fig. 3 Variation of the first modal eigenvalue with effective aspect ratio.

neously equal to zero correspond to the desired eigenvalues for this coupled problem.

One of the problems with the computational model described is that when the coupling parameter L_2 becomes identically zero, the coupled system of equations becomes computationally ill-conditioned and unsolvable. To circumvent this, the results for the uncoupled case ($\bar{\lambda}_c/\lambda_c = 1$) were obtained from a series of calculations using successively smaller values for the coupling parameter L_2. This led us to accept the values of eigenvalues for zero coupling as the value corresponding to the limit as the coupling approaches zero. Although this continuity assumption seems to make sense, it is not backed by a rigorous mathematical proof. Luckily it was possible to check these results with results generated for an isotropic/metal aluminum/wing by MIT,[3,4] and the agreement was found to be good for the cases checked.

Results and Discussions

The natural frequencies ω_n for the coupled bending-torsion oscillations for a composite aircraft wing in the presence of elastic coupling and warping restraint is found [as shown by Eq. (27)] to be a function of the ratio $(D_{22}/\rho)^{1/2}$, the length ℓ_0, and the nondimensionalized frequency parameter $\bar{k}_n$ for the wing. In this problem $\bar{k}_n$ is a function of only two parameters, i.e., λ_c, which may be considered as an effective nondimensionalized aspect ratio, and $\bar{\lambda}_c/\lambda_c$, which in a way measures the amount of elastic coupling in the wing ($\bar{\lambda}_c/\lambda_c = 1$ for zero coupling). It is, therefore, seen from Eq. (27) that in order to increase ω_n one needs to make ℓ_0 as small as possible and/or make (D_{22}/ρ) and $\bar{k}_n$ as large as possible. Such an exercise may be necessary when a tailoring of the frequency is needed to avoid instabilities (e.g., very low structural frequencies may provide an atmosphere for a coupling between the flexible modes and rigid-body motions, which in turn has a potential to result in instability). This kind of tailoring is made convenient through the use of Eq. (27) in which, for a particular wing configuration and composite material, every variable in the equation shall be known except $\bar{k}_n$ and D_{22}. Obviously, if we want high ω_n, as we said earlier, D_{22} should be made to be as high as possible (and of course ρ as low as possible). Once this is done the only other parameter to be tailored is $\bar{k}_n$.

The plot of $\bar{k}_1$ as a function of λ_c and $\bar{\lambda}_c$ is shown in Fig. 3 for all configurations having low camber to twisting coupling and all values of bending to twisting coupling. The results computed at MIT,[3,4] which were used to verify the present results, are shown in Fig. 3 as well. The important trend made visible in this investigation, as shown in Fig. 3, is that $\bar{k}_1$ (and hence ω_1) decreases with increasing nondimensionalized coupling L_2, which perhaps may be a more effective way to actually measure and compare elastic couplings D_{16} and D_{26}. For example, the results from MIT,[3,4] which were computed for some representative configurations, in dimensionalized form seem to represent systems with a fairly significant variation in coupling D_{26} or D_{16} (depending on the coordinate system). However, when nondimensionalized, the results shown in Fig. 3 seem to show little variation in coupling. In fact, they appear to be so close to the zero-coupling ($\bar{\lambda}_v/\lambda_c = 1$) or isotropic (or metal) case, that $\bar{k}_1$ for a metal or isotropic wing should be a good approximation (if it was necessary to make an approximation). The low value for the effective coupling was also evident in the nondimensionalized results from MIT,[3,4] where the bending frequency hardly varied with material changes. The question that could be asked is, therefore, "Do all possible composite wing configurations result in very low effective nondimensionalized coupling, ($\bar{\lambda}_c/\lambda_c \approx 1$)?" If the answer is "yes," then it may be proposed that for $\lambda_c > 0.5$, $\bar{k}_1(\lambda_c,\bar{\lambda}_c/\lambda_c)$ may be approximated as $\bar{k}_1(\lambda_c,1)$, which is also the isotropic or metal value. For this case it is seen that the computation of the natural frequencies of composite aircraft wings with aeroelastic oscillations merely requires the computation of (D_{22}/ρ) for a given wing half-span ℓ_0, since $\bar{k}_n(\lambda_c,\bar{\lambda}_c/\lambda_c)$ is approximately equal to $\bar{k}_n(\lambda_c,l)$, which is approximately equal to a constant (3.5) for $n = 1$. This result should make frequency computations for the bending mode significantly easier.

The problem with an affirmative answer, which may likely be a "practical" answer, to the question posed is that there does not seem to be a theoretical or rigorous analytical reason (to the best of the authors' knowledge) why $\bar{\lambda}_c/\lambda_c$ must always be approximately 1. Therefore, if the answer to our question is negative, then the following observations may be made: 1) Significant variation in $\bar{k}_1$ is possible with variations in effective nondimensionalized coupling ($\bar{\lambda}_c/\lambda_c$). In fact, it can be seen from Fig. 3 that if $\bar{\lambda}_c/\lambda_c$ approaches zero, k_1 (and hence ω_1) approaches zero. 2) The values of $\bar{k}_1$ vary significantly with λ_c for low λ_c but approach asymptotic values for large λ_c. 3) The highest values of $\bar{k}_1$ is for isotropic (metals) or quasi-isotropic configurations. 4) For large values of λ_c, there appears to be a simple, approximate (hopefully linear) relationship between $\bar{k}_1$ and $\bar{\lambda}_c/\lambda_c$ (or a measure of coupling). 5) For very large coupling ($\bar{\lambda}_c/\lambda_c \to 0$), $\bar{k}_1$ approaches zero, which may provide the ingredient necessary for coupling between the elastic and rigid motions.

Perhaps a number of implications of some of these observations should be examined. Observation 3 seems to imply that the highest first frequency would correspond to isotropic or quasi-isotropic configurations if (D_{22}/ρ) and ℓ_0 are the same. It is known, however, that the metals have lower values of (D_{22}/ρ) than composites. It therefore means that quasi-isotropic or orthotropic configurations are desirable for such a design goal. Observation 5 would seem to imply that if a designer, interested in tailoring the wing frequencies, arbitrarily introduces large effective nondimensionalized coupling ($\bar{\lambda}_c\lambda_c \to 0$), then $\bar{k}_1$ (and hence ω_1) would approach zero. This may result in coupling between flexible and rigid motions, which may or may not lead to instabilities. If that is the case, is there an alternative, equivalent design without any penalties (weight or otherwise) that could have been explored? Although the answers to these questions can, strictly speaking, only be possible after carrying out the necessary aeroelastic analysis in which unsteady aerodynamic forces are considered, it appears from Fig. 3 that a rough idea of the final picture may be obtained from the natural frequency analysis. After all, it is a common belief that the phenomena that actually lead to aeroelastic instabilities are linked to damping and coupling.

Modal Transformation

Earlier in this paper it was mentioned that previous studies by other investigators have found what appeared to be some kind of modal transformations as ply orientation was changed in a design process for a composite wing. While variation in ply orientation may change several directional stiffness parameters for the wing, the coupling stiffness parameter L_2 may be singled out as a significant design parameter because it may vary considerably; orthotropic configurations have zero values, but it may be fairly significant in other configurations. Furthermore, it should be remembered that the main reason for ply orientation variation is for "tailoring," which is believed to be primarily tied to couplings (D_{16} and D_{26}). The absence or presence of these couplings is basically what differentiates orthotropic configurations from anisotropic configurations. From these observations, and the fact that the entity that ties the bending and torsional equations is the coupling, it became clear that the role of coupling in modal transformations should be significant.

In order to see the role of coupling in this study, the modal assumptions for the coupled problem were made similar to those normally made for the uncoupled problem (e.g., the frequency was assumed to be real) so as to provide an opportunity to compare, contrast, and discern the final results easily. When this was done and the eigenproblem was formulated, resulting in a complex transcendental expression from which the eigenvalues are to be extracted, a careful examination began.

A significant difference between the coupled and uncoupled problems was that (as shown in Fig. 2) the transcendental expression from which the eigenvalues are extracted is complex for the coupled problem while it is real for uncoupled problems. The complex nature of this transcendental expression basically reflects the fact that the bending and twisting oscillations are generally out of phase. Therefore, the resultant coupled frequency that represents both the bending and twisting oscillation may be viewed as some kind of vector representation of the individual contributions. In order to formulate some explanations for the phenomenon of modal transformation in coupled (conservative) systems, it may be necessary to compare and contrast coupled systems with damped systems. Damped systems by definition are nonconservative; i.e., the system experiences a net loss or gain in energy. It is well known that in a damped system the transcendental expression for extracting the eigenvalues is complex, again due to the phase angle that exists between the damping force and the conservative forces in the system. It is also known that some desirable types of damping would tend to reduce the oscillation of the system; the nondesirable types tend to make the oscillations diverge. Therefore, since damping is linked to some energy transfer that in turn tends to lead to a change in the oscillation frequencies, it was thought that the complex nature common to the coupled and damped system determinants (transcendental functions) from which the eigenvalues are extracted may be a similarity that may provide some explanations to the modal transformations in coupled systems. Using the similarity argument, the coupled system, which for the present problem is conservative, may be viewed as having some conservative intermodal energy transfer within the system when the coupling is changed, resulting in steady-state changes or transformations of the modal energy content of a coupled mode compared to the uncoupled case. It may be worthwhile to point out that some results recently obtained at Purdue University[8] and communicated to the authors seem to support this hypothesis strongly.

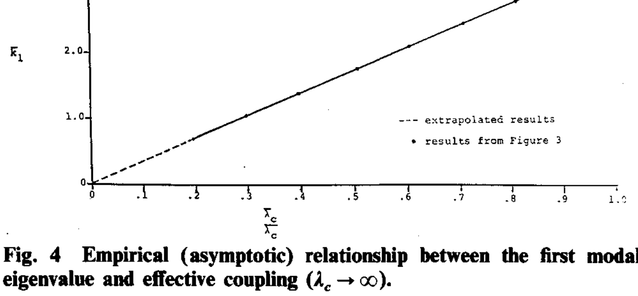

Fig. 4 Empirical (asymptotic) relationship between the first modal eigenvalue and effective coupling ($\lambda_c \to \infty$).

Empirical Relations

A careful study of Fig. 3 has led the authors to propose the following closed-form asymptotic relationship that may be useful for some preliminary design consideration:

$$\bar{k}_1 = 3.5(\bar{\lambda}_c/\lambda_c), \qquad \lambda_c > 3.0 \tag{28}$$

Equation (28) was derived from Fig. 4. Equation (28) as well as Eq. (27) show that the first coupled frequency decreases with increasing coupling, a trend that seems to be supported by new results from Purdue University[8] and the data from MIT.[3] In Ref. 3, e.g., the first nondimensionalized frequency computed by Rayleigh-Ritz (in which coupling is zero) had a value of 3.52, which is consistently higher than those computed by the finite-element method in which coupling is finite (not equal to zero). Equations (27) and (29), which are closed-form (generally rare for anisotropic systems), should be easy to use.

Before this discussion is concluded, it is probably necessary to explain why only the results of the first mode are shown in this paper. First of all it should be pointed out that some second-mode data have been generated but are still being studied very critically to understand the general trends. It may also be pointed out that the extraction of the eigenvalues is a little challenging, since some care is needed in ordering the roots of the operator equations.

Concluding Remarks

This paper has attempted to present exact closed-form solutions to the coupled bending-torsion vibration problem for a simplified model of composite aircraft wings with warping constraint. Increasing the coupling was found to decrease the first coupled frequency. A comparison between the coupled problem and a sample damped problem led the authors to propose some explanation to the "modal transformation" phenomenon found by earlier investigators. Some simplified closed-form expressions are provided for the first coupled frequencies that may be useful for fast applications.

Acknowledgments

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